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H.A. LAUWERIER

CHAOS AND ORDER

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Chaos and order *)

by

H.A. Lauwerier

ABSTRACT

The concept of "chaos" considered here is the apparent stochastic behaviour of deterministic dynamical systems, such as for example a system of ordinary differential equations or an iterative planar map. The last ten years the research in this field is rather explosive. Also due to computer experiments, important results are obtained by both mathematicians and physicists. An example is the period-doubling of Feigenbaum and the so-called KAM - theorem. With examples from biology and experimental physics we give in an illustrative way an impression of the fascinating and complicated aspects of this field. Topics are turbulent flows of a fluid, the Lorenz attractor, celestial mechanics, the KAM - theorem, self-similarity of strange attractors, the Hénon attractor, the map $x_{n+1} = ax_n(1-x_n)$, $0 < a \leq 4$, Feigenbaum's constant, the Julia theory, chaotic behaviour of analytic functions, variations on a biological theme.

KEY WORDS & PHRASES: *chaotic behaviour*, *dynamical systems*, *strange attractors*, *Feigenbaum's constant*, KAM - *theorem*.

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CHAOS AND ORDER[1)]

H.A. Lauwerier

INTRODUCTION

In this talk I wish to draw your attention to some exciting recent discoveries in the rapidly expanding field of non-linear analysis. The central theme is the apparently chaotic behaviour of deterministic dynamical systems with such characteristic features as strange attractors, self-similarity and fractals. These phenomena can be observed already in very simple mathematical models. In fact, general ideas and principles will be illustrated here by a number of well-chosen examples. The illustrations accompanying them were all obtained in an independent way by the help of a personal computer with an external plotter[2)]. By nature they have a single defect. As s t a t i c pictures they are poor representation of d y n a m i c processes. Any reader who has access to a personal computer or to a computer terminal is strongly invited to produce similar pictures by himself thereby adding a dimension of time. It is fascinating and inspiring to imitate the experiments of the great pioneers Hénon, Lorenz, Feigenbaum and others.

We are witnessing today the rise of a new discipline, experimental mathematics. It has its roots in the remote past. As we all know the inspection of tables of integers, squares and prime numbers inspired great mathematicians to intriguing conjectures and famous discoveries. But now the computer technology has given us a tool with undreamed-of possibilities. In the hands of able mathematicians it can be an invaluable instrument in a continuous interplay of theory and experiment, of conjecture proof and disproof.

1) English written version of a talk given for the Dutch Mathematical Congress. Delft April 6, 1983.

2) We used a HP 85.

Perhaps the most famous recent discovery, the universal period-doubling sequence, was made by Feigenbaum using a simple programmable pocket calculator. A schoolboy or a beginning student could have made the same discovery!

Turbulent flow of a fluid

Already it has given physics new insight into the nature of turbulence, irregular fluid motion, the onset of chaos as they call it.

Fluid motion can be described by a set of partial differential equations of the following kind

$$(1) \qquad \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\, \vec{u} = - \nabla p + \nu\, \Delta \vec{u}$$

where $\vec{u}(u,v,w)$ is the velocity vector at the position x,y,z and at time t, where p is the hydrostatic pressure and ν the viscosity. These equations can be solved only in very special cases. One such case models laminar flow in an infinite channel $-\infty < x < \infty$, $0 < y < 1$

$$(2) \qquad u = w = 0, \quad v = r\, y\, (1-y).$$

This is the so-called Poiseuille flow. In 1883 Reynolds found that theory and experiment were in good agreement only for moderate values of r but that beyond a certain threshold value the actual flow pattern showed the forming of vortices resulting eventually in a completely chaotic behaviour, so-called turbulent flow. In a more general set-up Reynolds introduced a dimensionless parameter $\ell\, v/\nu$, the so-called *Reynolds number* where ℓ is a typical length, v a typical velocity and ν the viscosity.

Attempts to understand turbulent behaviour on the basis of a set of differential equations such as (1) have been unsuccessful until recent times. In 1963 the physicist and meteorologist E.N. Lorenz published a simplified model of fluid flow which showed some sort of stochastic motion [3].

"... A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic. The feasibility of very-long-range weather prediction is examined in the light of the results".

His model is the following autonomous system of o.d.e.'s

$$(3) \qquad \begin{cases} \dot{x} = -\sigma x + \sigma y , \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy . \end{cases}$$

The parameter r has the meaning of the Reynolds number. The other two parameters are fixed and in Lorenz' paper taken as $\sigma = 10$ and $b = 8/3$.

According to the standard theory this system has a stable equilibrium up to the critical Reynold number $r = 24.74$. Lorenz considers the slightly supercritical case $r = 28$. It appears that, in modern language, within a certain region all orbits are attracted towards a strange attractor - the now famous Lorenz attractor -. All orbits look the same as shown in fig. 1 in a $(x+y,z)$ - projection. Lorenz noticed extreme sensitivity with respect to small perturbations of the initial values. This shows that for practical reasons the position of a phase point (x,y,z) in course of time is as unpredictable as the weather on the long run. This in spite of the fact that the system (3) is fully deterministic and that (x,y,z) for all times is uniquely determined by initial conditions at $t = 0$.

Although this paper remained unnoticed for several years in the mathematical community the Lorenz attractor has been and still is an object of much research [21], [25].

The Lorenz attractor seems to fill the pages of a half-open book. A cross-section through its leaves has the appearance of a Cantor set. Thus the structure has already been nicknamed a Cantor book. The work of E.N. Lorenz is a typical example of experimental mathematics made possible by the availability of an electronic computer. It was soon followed by a purely theoretical paper of equal historical interest (1971). I only quote the authors and the title: the mathematician F. Takens and the physicist D. Ruelle who wrote on "the nature of turbulence" [6], [16]. In this paper the notion "strange attractor" was used for the first time. Quoting Ruelle "... the name is beautiful, and well suited to these astonishing objects, of which we understand so little".

Usually a strange attractor is characterised by properties as self-similarity and a Cantor-type cross-section. However, it is not a properly defined mathematical object.

Celestial mechanics

Chaotic motion is as old as the world. This was realised by H. Poincaré who tried to solve a prize question proposed in 1885 by King Oscar II of Sweden. "For an arbitrary system of mass points which attract each other according to Newton's laws, assuming that no two points ever collide, give the coordinates of the individual points for all time as the sum of a uniformly convergent series whose terms are made up of known functions".

The prize was awarded to Poincaré although he did not in fact solve the problem. Poincaré noticed that even simple dynamical systems with two degrees of freedom, e.g. a restricted three-body problem such as an asteroid circling the sun under perturbation of Jupiter, could show very complicated behaviour. Such a motion can be represented by an orbit in a four-dimensional phase space (two dimensions for the coordinates and two dimensions for the momenta). However, since the total energy is an invariant of the system the motion in phase space is actually three-dimensional.

This means that the motion can be described by an autonomous system like (3). The unperturbed motion, say without Jupiter, is characterised by the trajectories in the x,y,z-space forming simple closed curves. So the fundamental problem is to analyse the effect of a small perturbation, say the presence of Jupiter. An orbit which was closed in its unperturbed state might be transformed into some sort of spiral situated on a toroidal surface but there are more possibilities. An effective way of studying such complications is the Poincaré map formed by the intersections of the trajectories with a suitable transverse two-dimensional surface. Let us assume for simplicity that the x,y - plane is such a surface of section. An unperturbed orbit which starts in an arbitrary point P_1 of this plane will return to the same position after a full revolution but if the orbit is perturbed the point of return P_2 will be different from P_1. This procedure defines an iterative map also called the return map. So the systematic study of dynamical systems with two degrees of freedom can be reduced eventually to the systematic study of planar iterative maps

$$(4) \qquad \begin{cases} x_{n+1} = f(x_n, y_n) \ , \\ y_{n+1} = g(x_n, y_n) \ . \end{cases}$$

Usually these mappings have the property of being area-preserving. If J denotes the Jacobian

$$(5) \qquad J = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

this corresponds to $|J| = 1$.

Maps of the kind (4) and their higher-dimensional generalizations are extremely important nowadays not only for modelling the motion of the celestial bodies but also the motion of elementary particles in the modern apparatus of high-energy physics. Moreover, they occur in the field of mathematical biology e.g. as models of the dynamic behaviour of populations. In the latter case the maps are as a rule not area-preserving and give rise to still other interesting phenomena.

The KAM - theorem

A most important theoretical break-through has been made by KOLMOGOROV (1954), ARNOLD (1961) and MOSER (1962) who building upon the work of Poincaré, Birkhoff and Siegel formulated a result which is now known as the KAM-theorem. According to this theorem after a small perturbation (say by Jupiter) of an integrable mechanical system such as an asteroid circling round the sun the majority of the perturbed orbits are quasi-periodic. This behaviour can also be described as the existence of a rather dense set of invariant tori in phase space, the so-called KAM-surfaces. An orbit which starts on such a surface stays there forever.

The KAM-theorem which is of great generality has far-reaching consequences. It explains for instance the absence of certain asteroid belts and the presence of many others in our solar system [4], [9],

[10], [27].

To an unperturbed orbit in phase space we may attach a winding number α. The KAM-theorem says that by a small perturbation all orbits are destroyed for which α is rational or close to rational. However, all orbits survive for which α is sufficiently irrational, i.e. for which there exists inequalities

$$(6) \qquad \left| \alpha - \frac{p}{q} \right| > \frac{C}{q^{2.5}}$$

where p/q is a rational approximation.

The KAM-theorem has been proved only for very small perturbations but the predicted behaviour is also observed experimentally for larger perturbations. Recent experiments [18], [24], [27] indicate that for increasing values of the perturbation parameter more and more KAM-surfaces break down until a single KAM-surface still stands up with a winding number $\alpha = \frac{1}{2}(1+\sqrt{5})$, the golden ratio and Fibonacci number. When eventually also this last KAM-surface is destroyed full-scale chaos sets in.

In order to illustrate the typical quasi-chaotic behaviour of dynamical systems we consider an iterative map (4) as the Poincaré map of a three-dimensional phase space. A closed orbit in 3-space then corresponds to a fixed point or to a periodic cycle of the planar map. A quasi-periodic orbit on an invariant torus in 3-space corresponds to invariant curves in the planar map. In 1969 the astronomer M. Hénon [5] performed a numerical study in which he considered the simplest map of type (4) where f and g are quadratic polynomials. Using the condition $J = 1$ the map could be reduced to the following simple form

$$(7) \qquad \begin{cases} x_{n+1} = x_n(\cos\alpha - (y_n - x_n^2) \sin\alpha, \\ y_{n+1} = x_n \sin\alpha + (y_n - x_n^2) \cos\alpha\,. \end{cases}$$

Fig. 2 shows a few orbits for the case $\cos\alpha = 0.24$. The origin, an elliptical fixed point, is surrounded by a few KAM-curves. We see that for a winding number with a 5-resonance the resonating orbit is broken up in a ring of 10 secondary fixed points, a stable periodic cycle of elliptic points and an unstable periodic cycle of hyperbolic points in an alternating order as predicted already by Birkhoff. Each secondary elliptic fixed point is surrounded by KAM-orbits and island chains. A secondary hyperbolic fixed point is surrounded by chaotic orbits as is shown in Fig. 3 in a blow-up of the vicinity of such a hyperbolic point. All points belong to the same orbit consisting of some 20.000 points. The chaotic region may spread all over the place but KAM-curves are impenetrable to chaotic orbits. Fig. 2 shows an outer stochastic ring as a single orbit which stays outside the outer KAM-curve of the picture. However, the feature of bounded chaos is a characteristic only of two-dimensional mappings. In higher-dimensional systems a chaotic orbit may get almost everywhere and still avoid all KAM-tori.

Self-similarity

Let us consider one of the five big islands of fig. 2 with respect to the five-fold iteration of the original map (7). The central elliptic periodic point now becomes an ordinary fixed point. The whole story repeats itself for this island, and of course for any other island the centre of which is an ordinary fixed point of some higher iterate of (7). In this way we are observing nesting sequences or trees of islands in a self-similar fashion.

In our days we are beginning to understand the laws of self-similarity. Experiments indicate the existence of universal laws which apply to large families of dynamical systems in one and the same way. Thus the behaviour of the area-preserving map of Hénon is generic and is perhaps typical for all sufficiently smooth area-preserving maps. The first great discovery has been made by M. Feigenbaum in 1975 but before discussing this most important topic we want to present a few other members of the family of planar area-preserving maps.

In recent years experimental physicists have spent much effort in the construction of accelerators and storage rings for carrying out experiments in which the collision of intersection beams of elementary particles might give birth to entirely new kinds of particles [9].

During such an experiment protons must orbit 10^{10} to 10^{11} times around a circular path and always be contained within a tunnel with a cross-section of postcard size. The collision of beams means a very slight perturbation that can be studied from a Poincaré map. Motivated in this way GUMOWSKI and MIRA [13] considered the two-step iteration

$$(8) \qquad x_{n+1} + x_{n-1} = 2\,F(x_n) \;,$$

where

$$(9) \qquad F(x) = cx + \frac{2(1-c)x^2}{1+x^2} \;.$$

The iteration (8) can be written as a planar map (4) as

$$(10) \qquad \begin{cases} x_{n+1} = y_n, \\ y_{n+1} = -\,x_n + 2\,F(y_n). \end{cases}$$

Fig. 4 gives an impression of the behaviour for the parameter value c = 0.25. The interpretation may be left to the reader.

A quite different source of iterative mappings showing possible chaotic behaviour can be found in numerical mathematics. The usual technique to adapt a continuous mathematical model to computer treatment is to replace derivatives by difference quotients and integrals by sums. In this way a differential equation is transformed into an iterative process. As we soon shall see already very simple problems may give rise to rather unexpected chaotic behaviour when a certain parameter, say the mesh width, exceeds some threshold.

Let us consider the differential equation of the pendulum

$$(11) \qquad \ddot{x} + \mu\,F(x) = 0$$

where e.g. $F(x) = \sin x$ or a similar non-linear function. For purposes of illustration we take

$$(12) \qquad F(x) = x\,\frac{1-x^2}{1+x^2} \;.$$

If the second derivative is replaced by a central difference, perhaps neglecting the advise of an expert, we obtain a two-step iteration

$$x(t+h) + x(t-h) - 2x(t) = -\,h^2\,\mu\,F(x)$$

written slightly different as

(13) $$x_{n+1} + x_{n-1} = 2x_n - \mu F(x_n)$$

and of the same form as (8). Again we obtain an area-preserving map

(14) $$\begin{cases} x_{n+1} = y_n \, , \\ y_{n+1} = - x_n + y_n - \mu F(y_n). \end{cases}$$

Fig. 5 shows the picture for the case $\mu = 3$. Conclusions are left to the reader.

The Hénon attractor

In 1976 M. Hénon made a further experiment [16]. His starting-point was to consider the "simplest" non-linear two-step recurrent relation

$$x_{n+1} = \alpha x_n + \beta x_n^2 + \gamma x_{n-1} \, .$$

With a bit of scaling this can be written as

(15) $$x_{n+1} + a x_n^2 - b x_{n-1} = 1$$

or in the form of a planar map as

(16) $$\begin{cases} x_{n+1} = y_n + 1 - ax_n^2 \, , \\ y_{n+1} = bx_n . \end{cases}$$

We note that the map is area-preserving for $|b| = 1$. For $|b| < 1$ the map is said to be area-contracting or dissipative. Hénon considers the particular case $a = 1.4$, $b = 0.3$ and finds that all bounded orbits have the same limit set, a noodle-like structure, a truly strange attractor, called nowadays the Hénon attractor.

A typical plot is given in fig. 6. In the following two figures blow-ups are given of the small square in the previous picture. Each time a similar structure of lines is obtained as another demonstration of the built-in self-similarity. This similarity across the various layers corresponds to the structure of a Cantor set. The many-leaved structure of the Hénon attractor can be characterised by its Hausdorff-dimension. Let S be a many-leaved structure as a subset of an N-dimensional square and let $M(\varepsilon)$ be the minimum number of N-dimensional cubes of side ε needed to cover S. Then

$$(17) \qquad d(S) = \lim_{\varepsilon \to 0} \frac{\log M(\varepsilon)}{\log (1/\varepsilon)}$$

defines the so-called fractal dimension or Hausdorff dimension of S. Typically for a strange attractor as the Hénon attractor d(S) is a non-integer number. The classical Cantor set obtained by deleting open middle thirds starting from the closed unit interval appears to have dimension $\log 2/\log 3 = 0.630$. A direct experimental calculation shows that the Hénon attractor has the fractal dimension 1.261 ± 0.003 [27], [28].

It is interesting to observe experimentally what happens for other values for a,b. For e.g. $a = 1.3$, $b = 0.3$ the attractor suddenly disappears and is replaced by a periodic cycle of seven points!

<u>The simplest non-linear iterative map</u>

In 1976 the biologist R.M. MAY [7] showed that chaotic behaviour could be observed already in very simple one-dimensional maps. The impact of his paper upon the scientific community can hardly be overestimated. It started new lines of research and although a bit outdated it is still highly recommendable reading, see also [11], [19], [20]. The simplest mapping is

$$(18) \qquad x_{n+1} = a\, x_n (1-x_n), \qquad 0 < a \le 4.$$

In population dynamics it models limited growth of ,say, insect populations of successive generations. The behaviour of the map (18) for $0 < a \le 3$ is quite regular. All orbits starting in [0,1] converge to either 0 ($a<1$) or $1-1/a$ ($a\ge 1$). However, if $a > 3$ the limit set can be a finite periodic cycle.

For $3 < a < 3.57$ there exists a sequence of values a_m, $m = 1,2,3\ldots$ such that for $a_m < a < a_{m+1}$ the limit set is a periodic cycle of 2^m points. In particular $1_1 = 3$, $a_2 = 1 + \sqrt{6} = 3.449$, $a_3 = 3.544$ and $a_\infty = 3.569946$. What happens for a between a_∞ and 4 is extremely complicated [31]. The dominant feature is that for most parameter values in this range the behaviour is aperiodic, i.e. chaotic. However, periodic regimes are imbedded in the interval $(a_\infty,4)$. The following amusing computer experiment gives a good impression of the very rich bifurcation behaviour. In fig. 9 we have fixed a to be one of 300 equally spaced values in the interval (2.9,3.9). For each a we take $x_0 = 0.5$ and plot x_n for $200 < n < 400$ on a vertical scale as an approximation of the limit set.

The most conspicuous feature is a window around $a = 3.83$ for which there exists a stable 3-cycle. Smaller windows with stable cycles of other low periods can also be observed [11], [20].

The mapping (18) has an explicit analytical solution only in the cases $a = 2$ and $a = 4$. For $a = 4$ the orbits can be parametrised as

(19) $$x_n = \sin^2 (2^n \theta\pi) , \quad 0 \le \theta < 1.$$

This shows that (18) is equivalent to the shift map $\theta \to 2\theta \pmod 1$. One can visualise this mapping that at each iteration step the expansion of θ as a binary fraction is shifted one position to the left. Clearly starting with an irrational value θ the resulting orbit is aperiodic and looks as random as the tossing of a coin. For a rational θ we obtain an unstable periodic orbit. When working on a computer with a limited precision of, say, 12 decimal positions this means that after some 40 iteration steps all information contained in the starting value has been lost.! This extreme sensitivity for initial conditions is typical for dynamical systems having strange attractors. A similar phenomenon can be observed for the Lorenz attractor. A computer movie set up by J.D. Farmer shows this in an almost dramatic way. A group of initial points, say twenty, starting very close together on the attractor is travelling like an illuminated night-train in the dark on a tortuous railroad track. However, after a good number of turns the train seems to be extended too much and splits into two parts going either way. Next, very soon the train is broken up completely into separate components travelling in an apparently uncorrelated way i.e. chaotically.

Feigenbaum's universal law of period-doubling

Perhaps the most exciting discovery in the past few years, the universality of period-doubling, was made by Mitchell Feigenbaum in November 1975, [12]. Let us consider (18) and the following table which contains the values of a for which the orbit with period 2^{m-1} loses stability and bifurcates into an orbit of period 2^m. It is instructive and not difficult to reproduce the table using a small programmable pocket calculator such as an HP 41.

a_1	3				
a_2	3.449499	$a_2 - a_1$	0.449499	$(a_2 - a_1)/(a_3 - a_2)$	4.752
a_3	3.544090	$a_3 - a_2$	94591	$(a_3 - a_2)/(a_4 - a_3)$	4.656
a_4	3.564407	$a_4 - a_3$	20317	$(a_4 - a_3)/(a_5 - a_4)$	4.668
a_5	3.568759	$a_5 - a_4$	4352	$(a_5 - a_4)/(a_6 - a_5)$	4.669
a_6	3.569692	$a_6 - a_5$	932	$(a_6 - a_5)/(a_7 - a_6)$	4.669
a_7	3.569891	$a_7 - a_6$	200	$(a_7 - a_6)/(a_8 - a_7)$	4.669
a_8	3.569934	$a_8 - a_7$	43		

Table 1. The genesis of Feigenbaum's δ

With Feigenbaum we observe that the sequence a_m is behaving asymptotically as a geometric progression

(20) $$a_m \approx a_\infty - \text{const.}\ \delta^{-m}$$

with, as given by Feigenbaum,

$$\delta = 4.669201660910\ldots\ .$$

Feigenbaum discovered that the same asymptotic ratio δ was found also in other maps given by quite different formulas so that the bifurcation behaviour described above appears to be universal in some sense.

The Feigenbaum scenario has by now been observed also in most current low dimensional dynamical systems including the Hénon map and the Lorenz equations. It is a break-through in physics where it has been observed already in experiments with liquid helium and with laser light. Feigenbaum's conjectures have been proved so far only very partially [11] , [10] and true understanding is still lacking.

The universality of the period-doubling behaviour can be illustrated by the following diagram

$$\begin{array}{ccccc} z & & \xrightarrow{\quad\quad f \quad\quad} & & f(z) \\ \lambda \downarrow & & & & \Big| \lambda \\ \lambda z & \xrightarrow{f} & f(\lambda z) & \xrightarrow{f} & * \end{array}$$

At the asterisk we obtain the functional equation

$$(21) \qquad f(f(\lambda z)) = \lambda\, f(z)$$

as suggested by Feigenbaum [12], cf. also [11] and [20].
With the further requirement

$$f(z) \text{ even}\,, \quad f(0) = 1$$

this equation has still a multitude of solutions but there seems to exist a single analytic solution

$$(22) \qquad f(z) = 1 - 1.52763\, z^2 + 0.104815 z^4 - 0.0267057\, z^6 + \ldots$$

with $\lambda = -\,0.3995$, another universal scaling parameter. The universal constant δ is characterised as the largest eigenvalue of the linear operator

$$(23) \qquad h(\cdot) \to \frac{1}{\lambda}\, h(f(\lambda\cdot)) + \frac{1}{\lambda}\, f'(f(\lambda\cdot))\, h(\lambda\cdot).$$

Almost nothing is known about uniqueness of (22). If Hilbert would have lived these days he would have taken it as one of his challenging problems! Recent work of J. Greene, R.S. Mackay and others [18], [24] show similar scaling phenomena in planar maps with further universal constants. Many interesting results have been obtained in an experimental way but the overall picture is that of many conjectures, little understanding and very few rigorous proofs.

The Julia theory

The modern development of the theory of iterative maps has its roots in the past. The works of Poincaré are widely known and many a modern result can be traced back to one of his papers. However, there is more. In 1919 G. Julia [1] gave a rather complete treatment of one-dimensional complex iterative maps

$$z_{n+1} = \phi\,(z_n) \tag{24}$$

where $\phi(z)$ is a quotient of two polynomials. His main result is the existence of an exceptional set, the so-called Julia set J, which is the closure of the set of all unstable periodic points of (24). J is an invariant manifold, a separatrix separating the attracting domains of the stable fixed points. The map (24) has a multi-valued inverse. For the inverse map J becomes an attractor, as a rule what is now called a strange attractor. Julia proves that J is densely covered by the preimages of an arbitrary point (with at most two exceptions) and shows that J is a perfect set of one of the following three types

1. all of $\mathbb{C}$.
2. linearly connected, a collection of Jordan arcs.
3. completely disconnected.

Usually J appears to be a curve without tangents, a fractal with the familiar feature of self-similarity. Although Julia was unable to make visible any such curve he pointed out the possible likeness to Helge von Koch's famous snow-flake curve [28]. Julia's work has been almost untouched for a fifty odd years partly due to his writing in French. However, his prize-winning mémoire [1] is still strongly recommendable for its clarity in style and considered as a forerunner to the modern line of research.

For the complex version of (18)

(25) $$z_{n+1} = a\, z_n(1-z_n)\ , \quad z_n \in \mathbb{C}$$

with $1 < a \leq 4$ this means that J is a fractal curve. For a = 2 when (25) is solved by

(26) $$z_n = \tfrac{1}{2}\,(1-\theta^{2^n})$$

J is the circle $|z - \frac{1}{2}| = \frac{1}{2}$. For a = 4 when (25) is parametrised by (19) J is the closed interval $0 \leq x \leq 1$, $y = 0$ taking $z = x + iy$. In all other cases J is nowhere differentiable. In particular for a = 3 the Julia set is shown in fig. 10. MANDELBROT [28] has given it the nick-name of "San Marco attractor" in view of its superficial similarity with the skyline of the famous Venetian cathedral when mirrored in the flooded San Marco place. Shapes like this can easily be obtained on a computer screen by iterating (25) backwards

(27) $$2z_n = 1 \pm \sqrt{1 - 4\, z_{n+1}/a}$$

each time taking one of the signs at random. If the unstable fixed point z = 0 is taken as the start all further preimages will be points of J eventually forming a dense covering.

Still more exotic fractal shapes can be found in very much the same way. The map

(28) $$z_{n+1} = (3z_n - z_n^3)/2$$

given by Julia determines a fractal curve with a dense covering of multiple points, loops in loops in loops in endless succession. Fig. 11 shows its shape in a similar approximation as in the previous picture. Even more strange is the Julia set of

(29) $$z_{n+1} = z_n - \frac{3z_n^2}{z_n^3 - 1}\ ,$$

the familiar Newton process for finding the roots of $z^3 - 1 = 0$. It is an old problem and it is also discussed in Julia's paper. However, its fractal shape is too complicated to give a nice picture.

"Chaotic" behaviour of analytic functions

It is possible to parametrise complex iterative maps by means of well-defined analytic functions. This may bring the treatment of such maps within the realm of the theory of complex analytic functions where powerful tools are available. Again the method goes back to the French school of classical analysis with Poincaré, Fatou and Julia as leaders. Let us consider again the complex mapping (25). Then there exists a uniquely defined entire function F(z) satisfying the so-called Poincaré equation

$$(30) \qquad F(az) = a\,F(z)(1-F(z))$$

with the supplementary conditions

$$F(0) = 0, \quad F'(0) = 1.$$

Clearly (25) can be parametrised by

$$(31) \qquad z_n = F(a^n \theta).$$

The asymptotic behaviour of F(z) on the positive real axis contains the complete description of the behaviour of the real iterative map (18) for *all* possible starts. If e.g. $a_m < a < a_{m+1}$ in the notation of table 1 $F(a^t)$ appears to be asymptotically periodic with the period 2^m. Its asymptotic shape is a composition of horizontal and vertical line segments as shown in fig. 12 in which $a = 3.5699$ and $20 < t < 21$.

Again we observe features of scaling and self-similarity à la Feigenbaum. It is a tempting idea to derive the Feigenbaum scenario of period-doubling in an independent way using techniques of the theory of complex analytic functions. As a byproduct it appears that the entire functions defined by (30) have a quite remarkable asymptotic behaviour [29].

Variations on a biological theme

Simple variations of the model (18) as suggested in biological application give rise to further interesting problems. In [15] and [22] the iterative process

$$x_{n+1} = ax_n (1-x_{n-1}) \tag{32}$$

is considered as another model of limited growth. In the corresponding map

$$\begin{cases} x_{n+1} = y_n , \\ y_{n+1} = ay_n(1-x_n) , \end{cases} \tag{33}$$

the fixed point $x = y = 1 - 1/a$ loses stability at $a = 2$. When this value is passed the map undergoes Hopf bifurcation.

As a further increases the Hopf cycle grows until $a = 2.271012$. For that value it becomes identical with the unstable manifold through the origin, a strange attractor with an infinity of folds with a common cusp at the origin. The attractor is shown in fig. 13. Folds are almost invisible at this scale but a blow up of a small region at the origin in fig. 14 shows a single loop and suggests the existence of a further loop.

The following related map, some sort of interpolation between (18) and (33), exhibits a quite different behaviour

$$\begin{cases} x_{n+1} = y_n , \\ y_{n+1} = ay_n (1-\tfrac{1}{2}x_n - \tfrac{1}{2} y_n) . \end{cases} \tag{34}$$

Still the fixed point $x = y = 1- 1/a$ is stable up to a certain critical value, $a = 3$, but for $3 < a < 3.6276$ we have no Hopf bifurcation but a stable four-cycle. This is the beginning of another Feigenbaum scenario in the spirit of (20).

We end this survey with, as a final example, a non-invertible planar map as a member of a different family of biomathematical models

$$\begin{cases} x_{n+1} = ax_n(1-x_n-y_n), \\ y_{n+1} = bx_n y_n . \end{cases} \tag{35}$$

It is a so-called predator-prey model, a discrete version of the continuous models introduced by Volterra, Lotka, Kolmogorov and many others. It has a rich bifurcation behaviour. As an impression of what may happen we give in fig. 15 an illustration of the case a = 3.5, b = 3.3 a rather chaotical strange attractor but still with a sort of structure that remains to be explored.

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DATA FOR FIGURES

fig. 1 Parameters $\sigma = 10$, $b = 8/3$, $r = 28$.
Scale -80,80,-10,70 in $x+y$, z projection.
Double - approximation numerical integration as indicated by the scheme

$$x_{n+1} = x_n + f(x_n)\,\Delta t,$$
$$x_{n+2} = x_{n+1} + f(x_{n+1})\Delta t,$$
$$x_{n+1} = \tfrac{1}{2}\,(x_n + x_{n+2}),$$

with $\Delta t = 0.004$.

fig. 2 $\cos\alpha = 0.24$
Scale -1.6, 1.6, -1.2, 1.2
The stochastic ring is obtained
from the start 0, 0.75.

fig. 3 Blowup of fig. 2 at
0.5696, 0.1622.
Scale 0.50, 0.64, 0.10, 0.20
20.000 points of orbit starting at -0.1518, 0.5796.

fig. 4 c = 0.25.
Scale -12,12, -9,9
Single stochastic orbit of 4000 points
starting from 0.2, 0.2.

fig. 5 $\mu = 3$.
Scale -10,10, -10,10.
KAM curves are obtained for
2,2 ; 3.51, 3.51 ; 3.8, 3.8.
The stochastic ringstructure starts from 1.2.

fig. 6 a = 1.4, b = 0.3.
Scale -1.5, 1.5, -0.45, 0.45.

fig. 7 Scale 0.54, 0.73, 0.15, 0.21.

fig. 8 Scale 0.621, 0.640, 0.185, 0.191.

fig. 9 Scale 2.9, 3.9, 0. 1
step 1/300
$x_{n+1} = ax_n(1-x_n)$
$x_0 = 1/2 \quad 200 \le n \le 400.$

fig. 10 1500 points of S. Marco attractor
a = 3. Scale -0.1, 1.1, -0.45, 0.45.

fig. 11 1500 points of Julia set of
$$x_{n+1} = (3x_n - x_n^3)/2$$
Scale -3.3, -2.25, 2.25

fig. 12 Graph of $F(a^t)$ where $F(z)$ is the entire function defined by
$F(az) = a\,F(z)(1-F(z))$,
$F(0) = 0,\ F'(0) = 1$,
for $a = 3.55,\ 8 \le t \le 16$.
Scale 8,16, 0,1.

fig. 13 Attractor of

$$x_{n+1} = y_n$$
$$y_{n+1} = ay_n(1-x_n)$$

for $a = 2.27$.

Scale -0.3, 1.3, -0.1, 1.1.

fig. 14 id. Blow-up 10x

Scale -0.03, 0.13, -0.01, 0.11.

fig. 15 id. Blow-up 1000x

Scale 0.04015, 0.04015.

fig. 16 2000 iterates of the prey-predator map

$$\begin{cases} x_{n+1} = ax_n(1-x_n-y_n), \\ y_{n+1} = bx_ny_n, \end{cases}$$

for $a = 3.5$, $b = 3.3$.

Scale 0, 0.8, 0, 0.8.

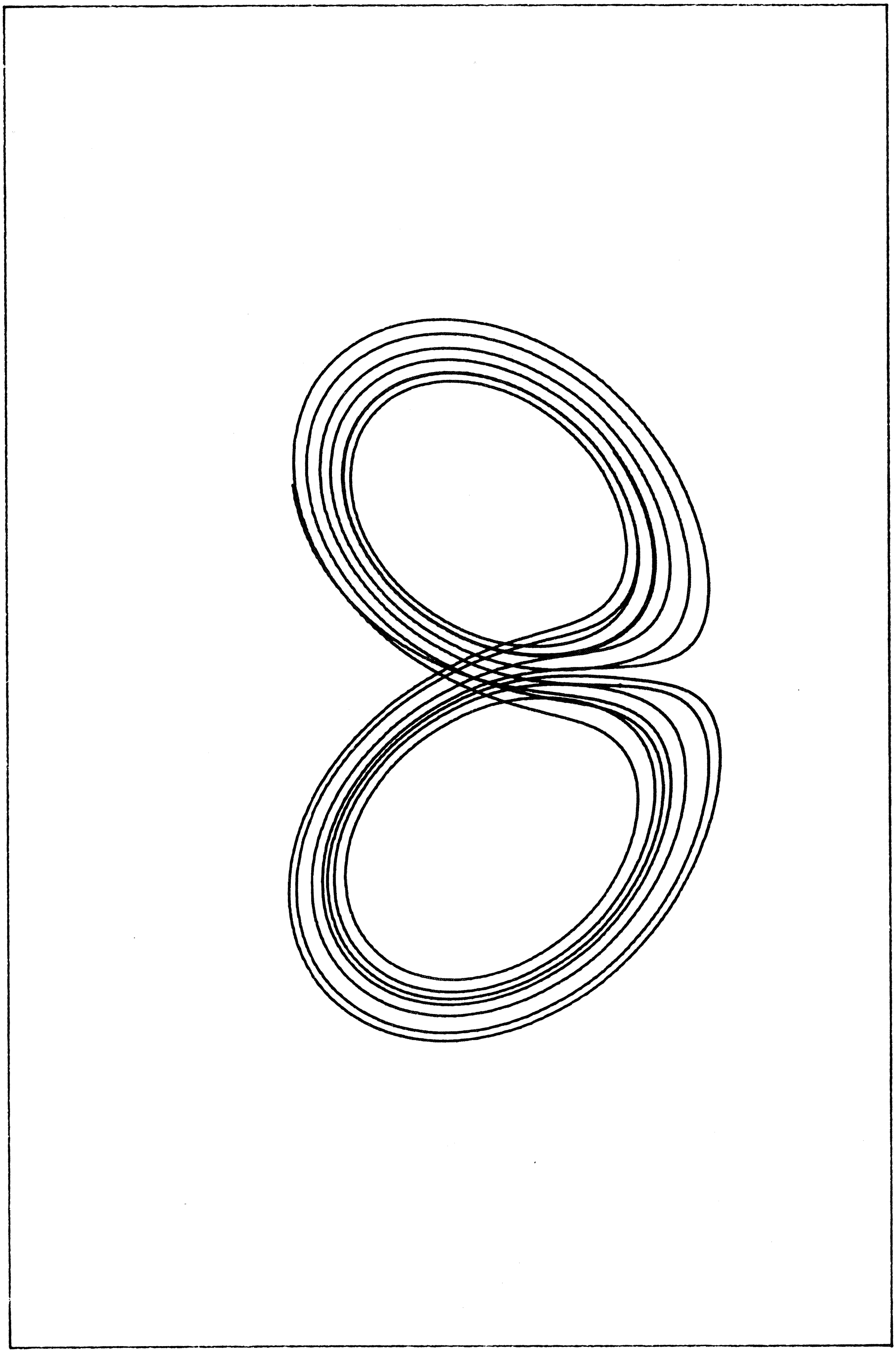

Figure 1. The Lorenz attractor.

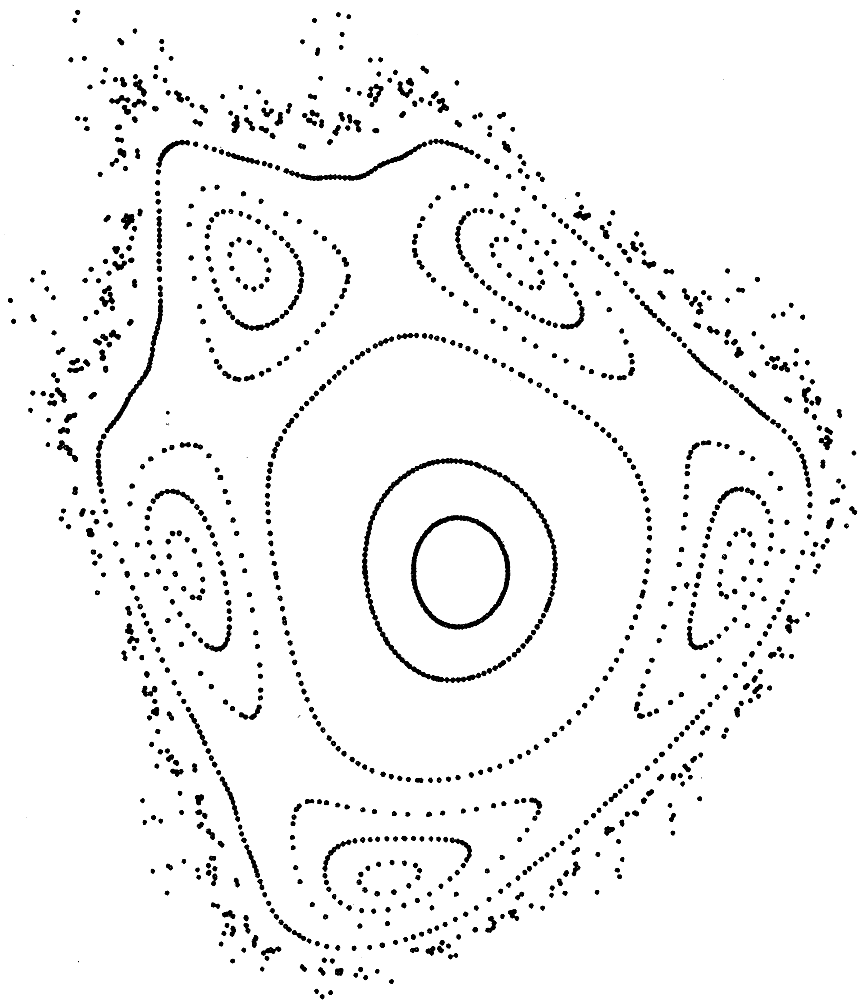

Figure 2. Area-preserving map of Hénon.
KAM - curves and stochastic ring.

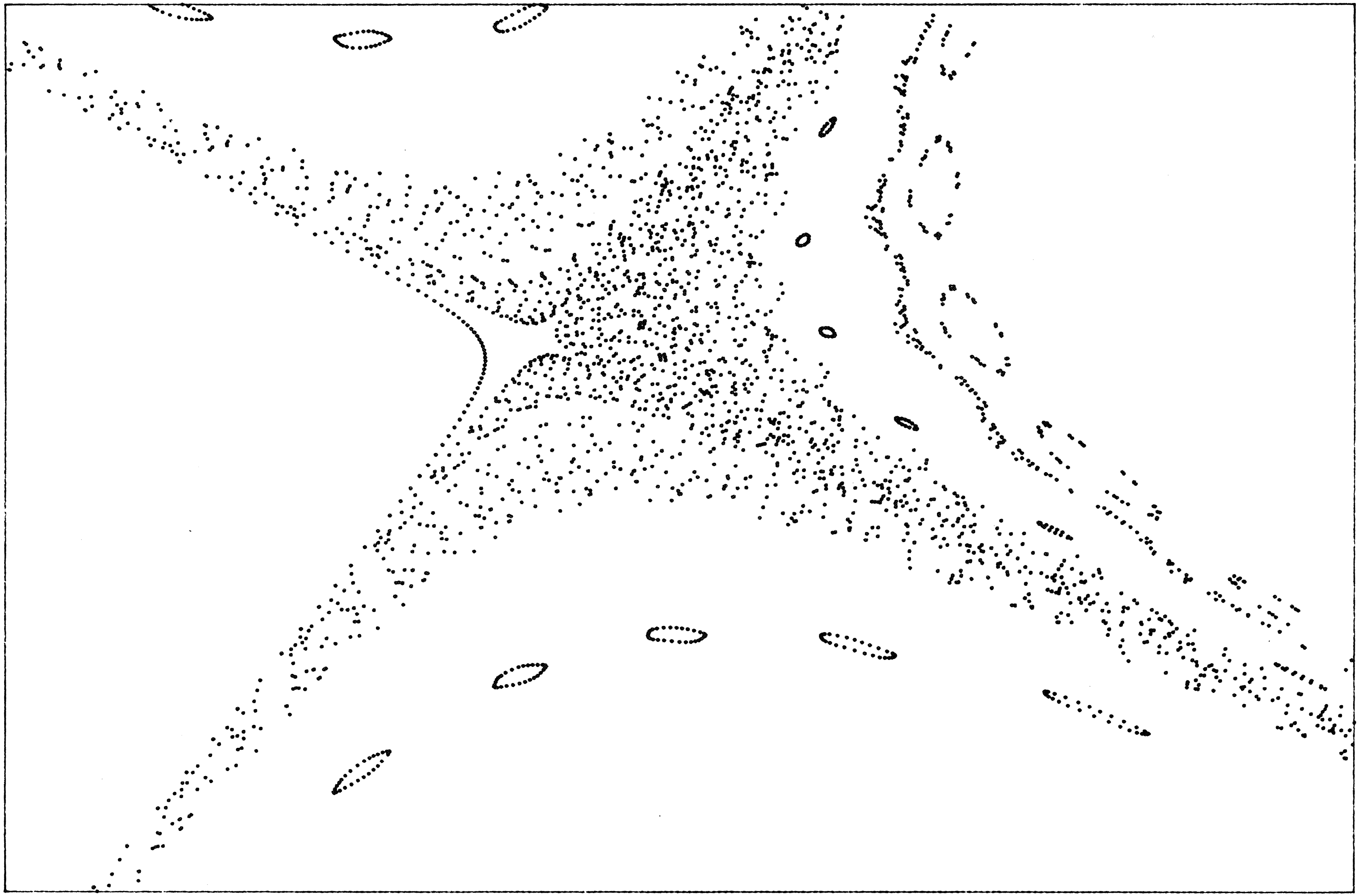

Figure 3. Blow-up of fig. 2 at a hyperbolic point. Stochastic orbit and island structures.

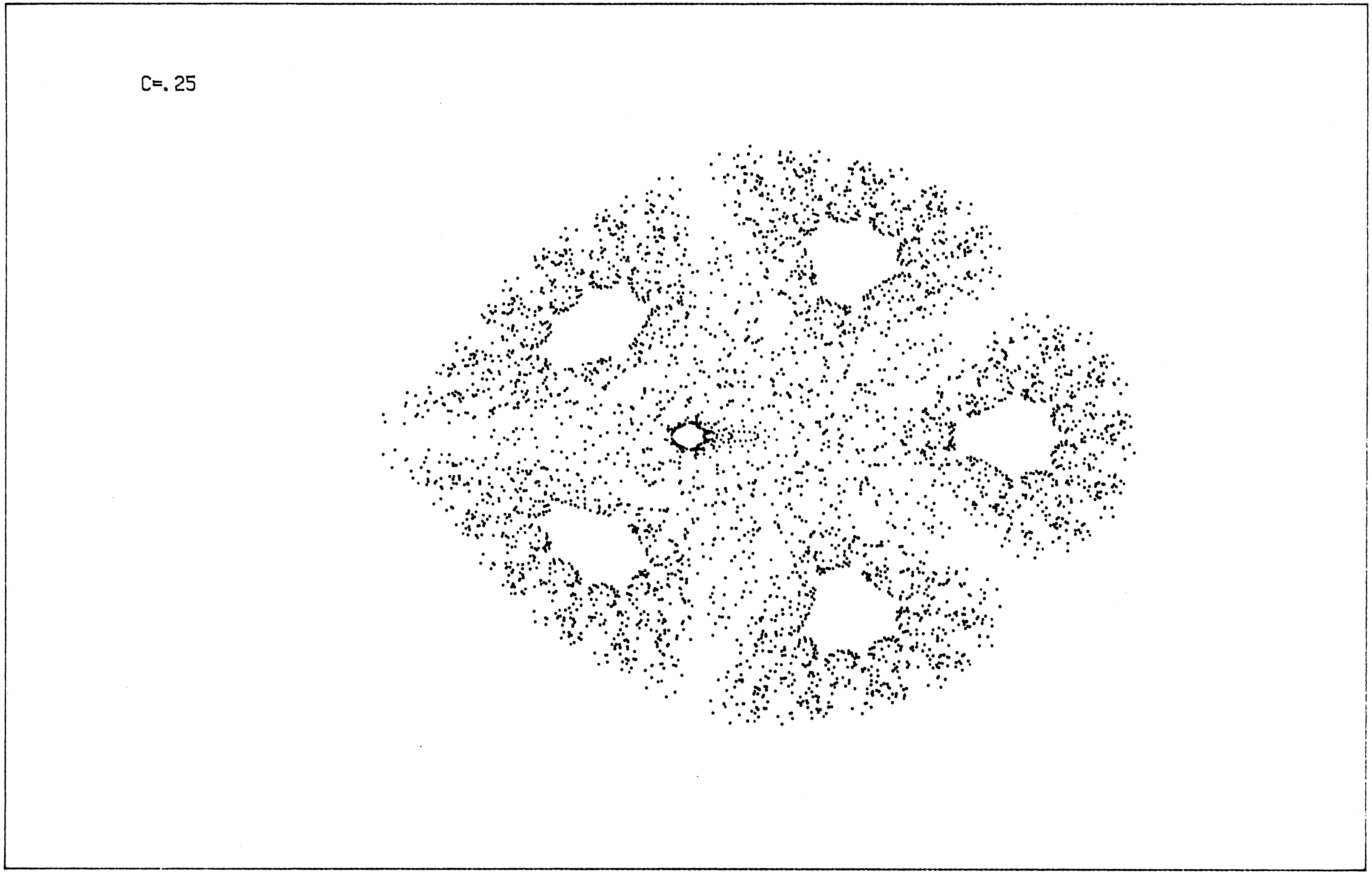

Figure 4. Area-preserving map as a model of the interaction of elementary particles. A stochastic orbit of 4000 points.

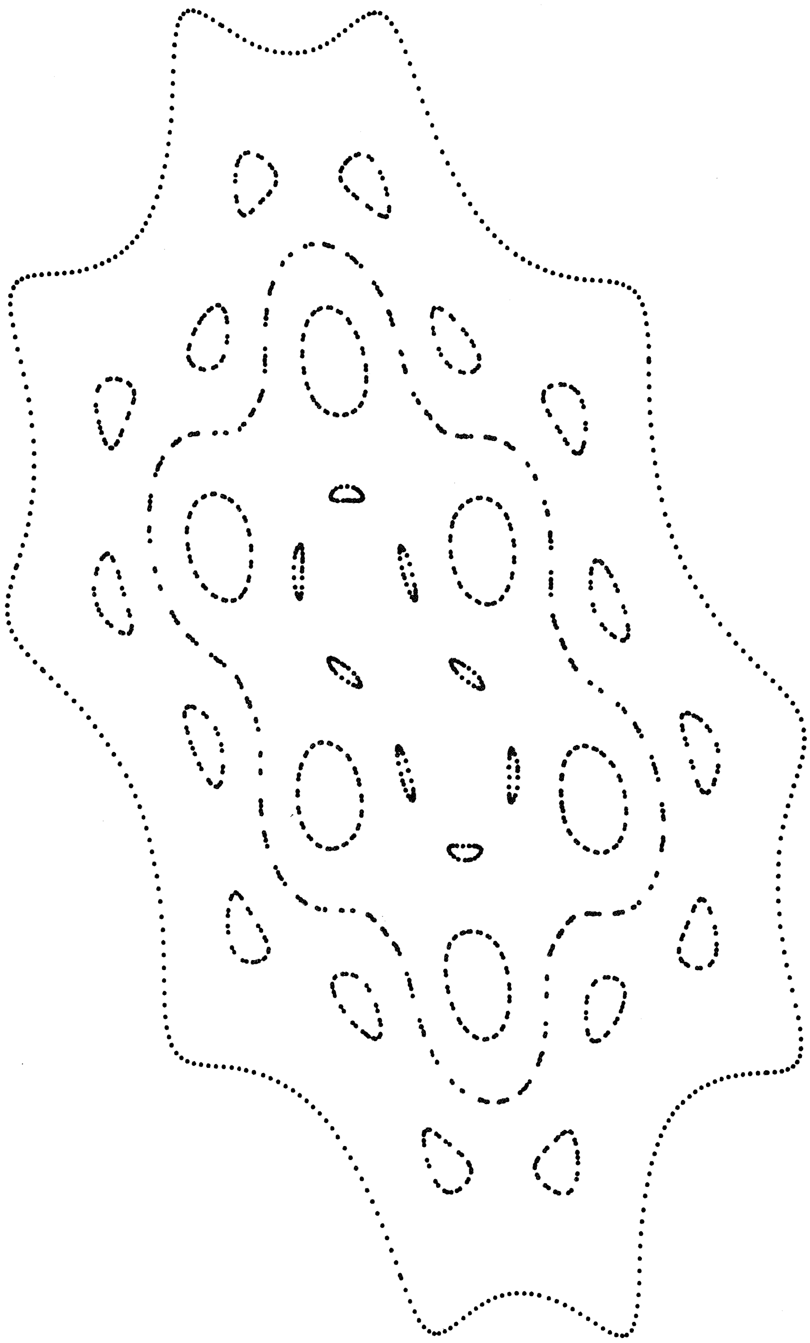

Figure 5. Area-preserving map in numerical mathematics.

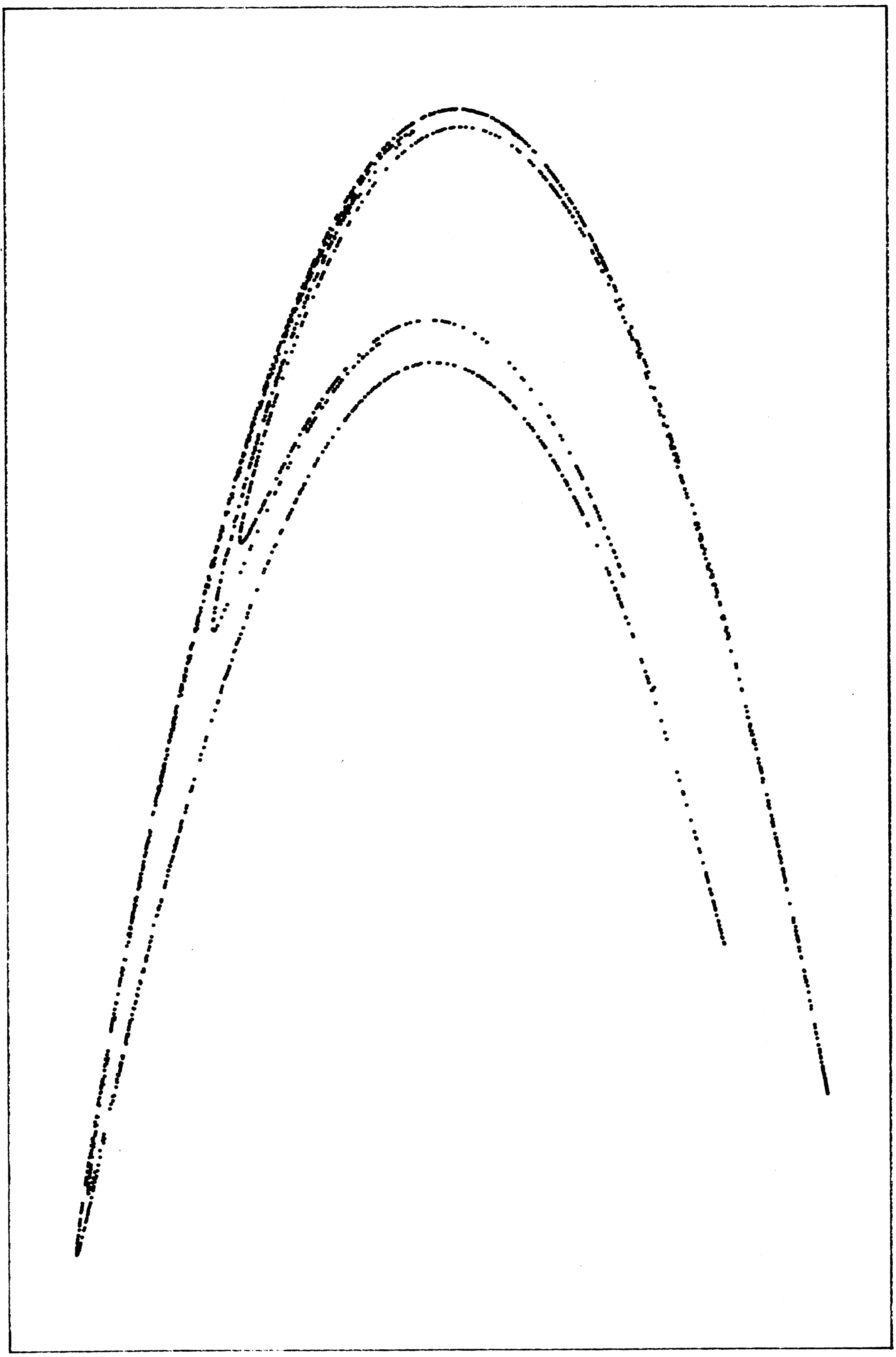

Figure 6. The Hénon attractor. 2000 points.

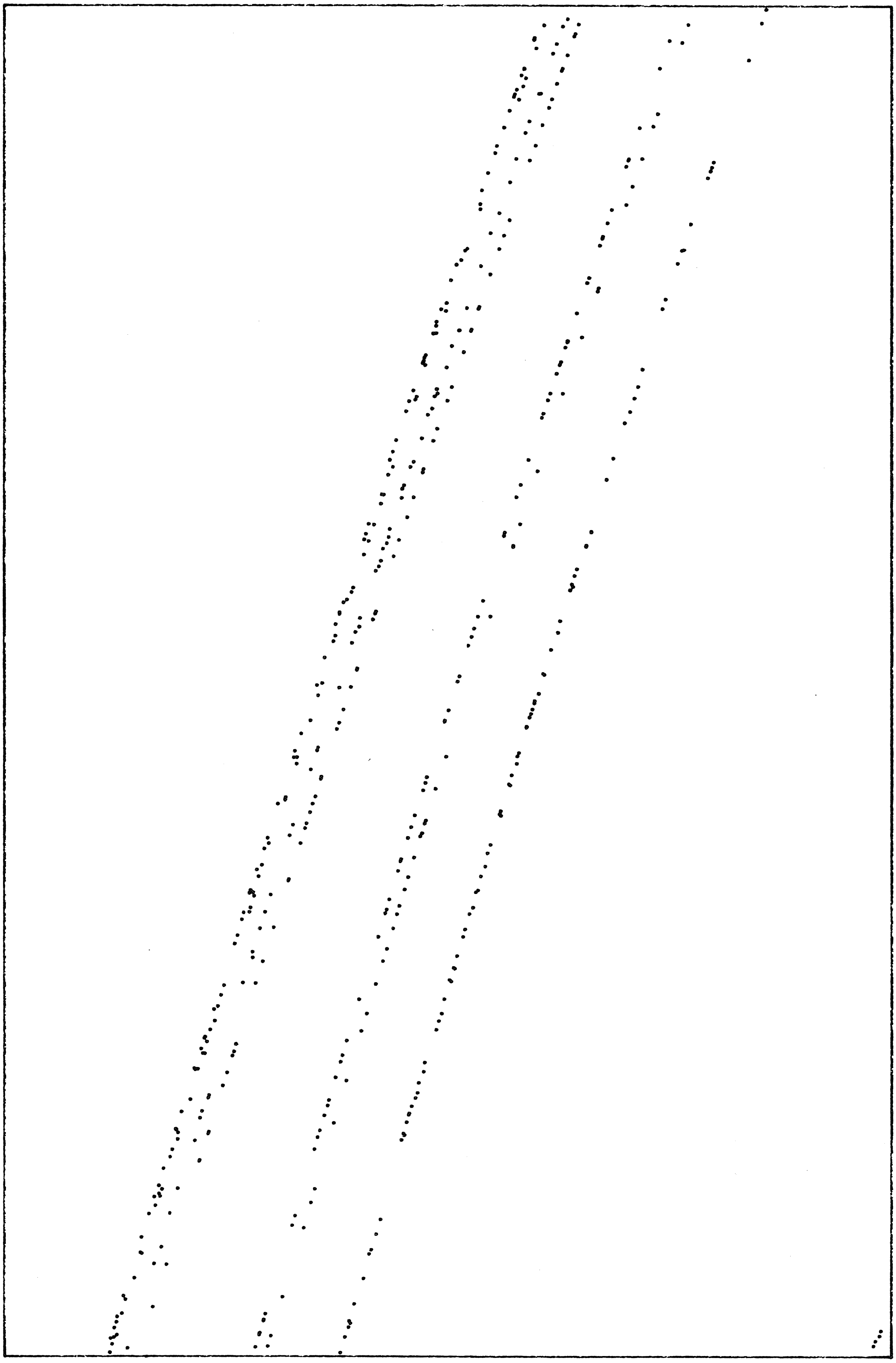

Figure 7. Blow-up of fig. 6

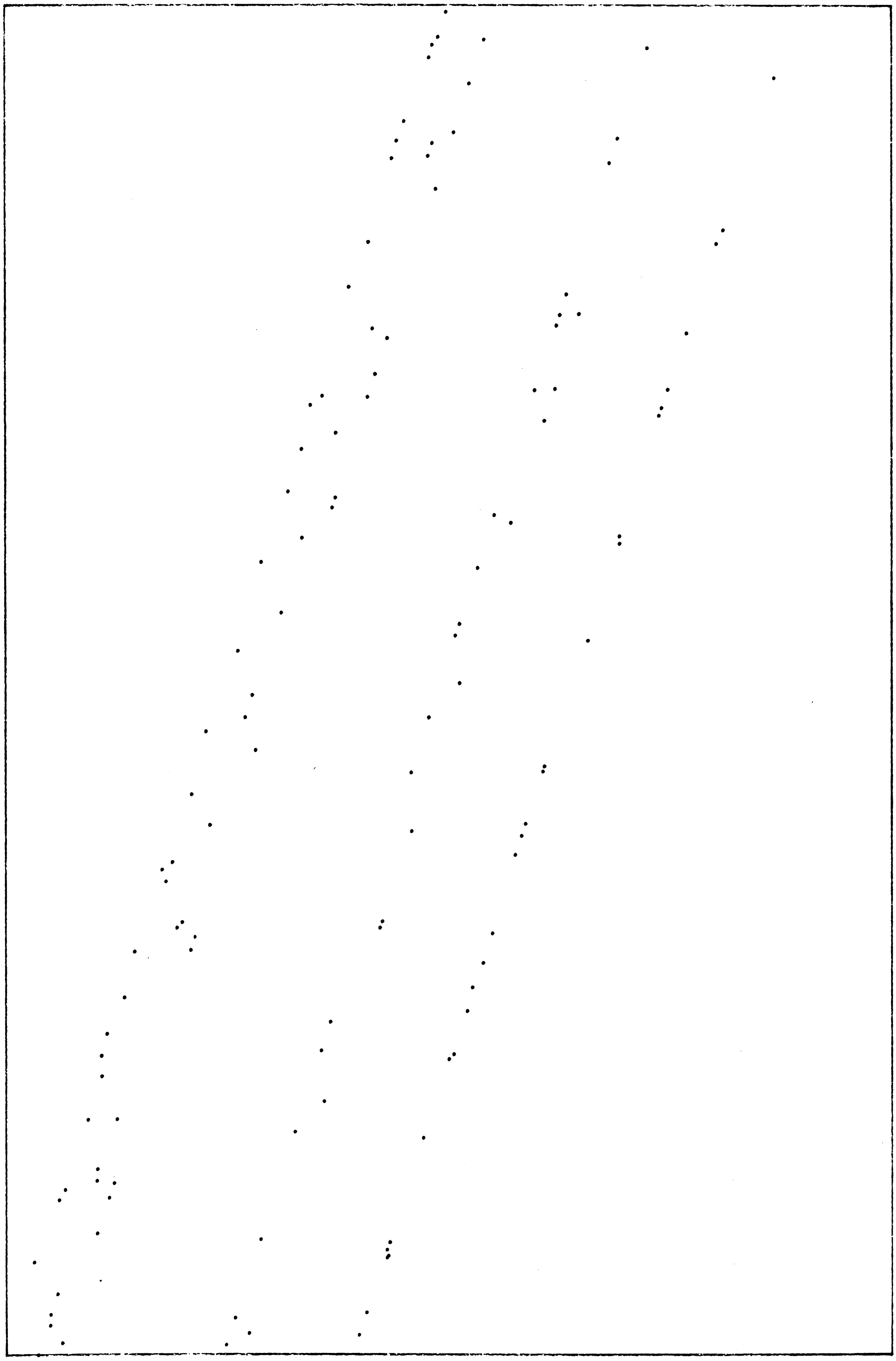

Figure 8. Blow-up of fig. 7.

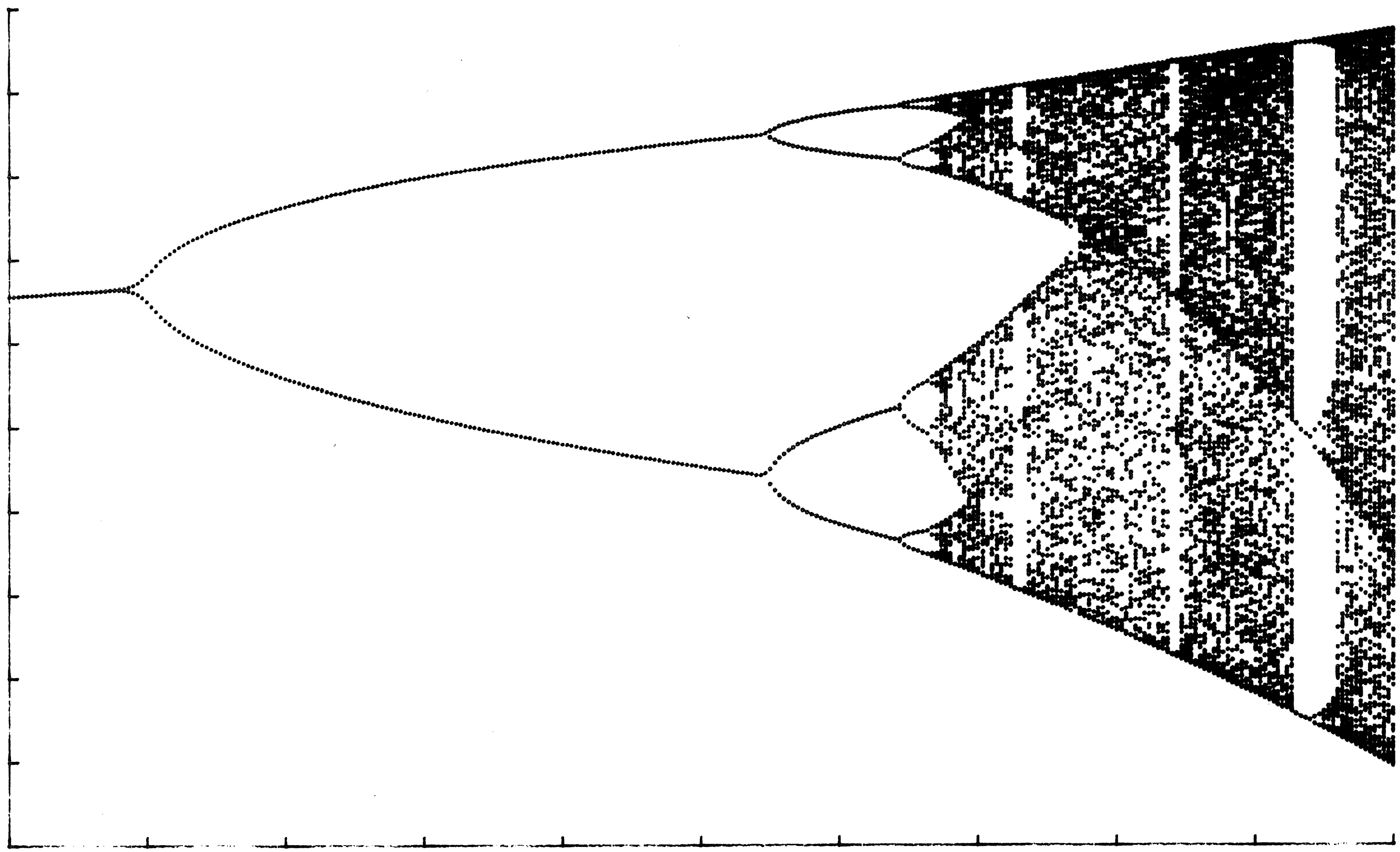

Figure 9. Bifurcation diagram of $x \to ax(1-x)$

For $2.9 \le a \le 3.9$

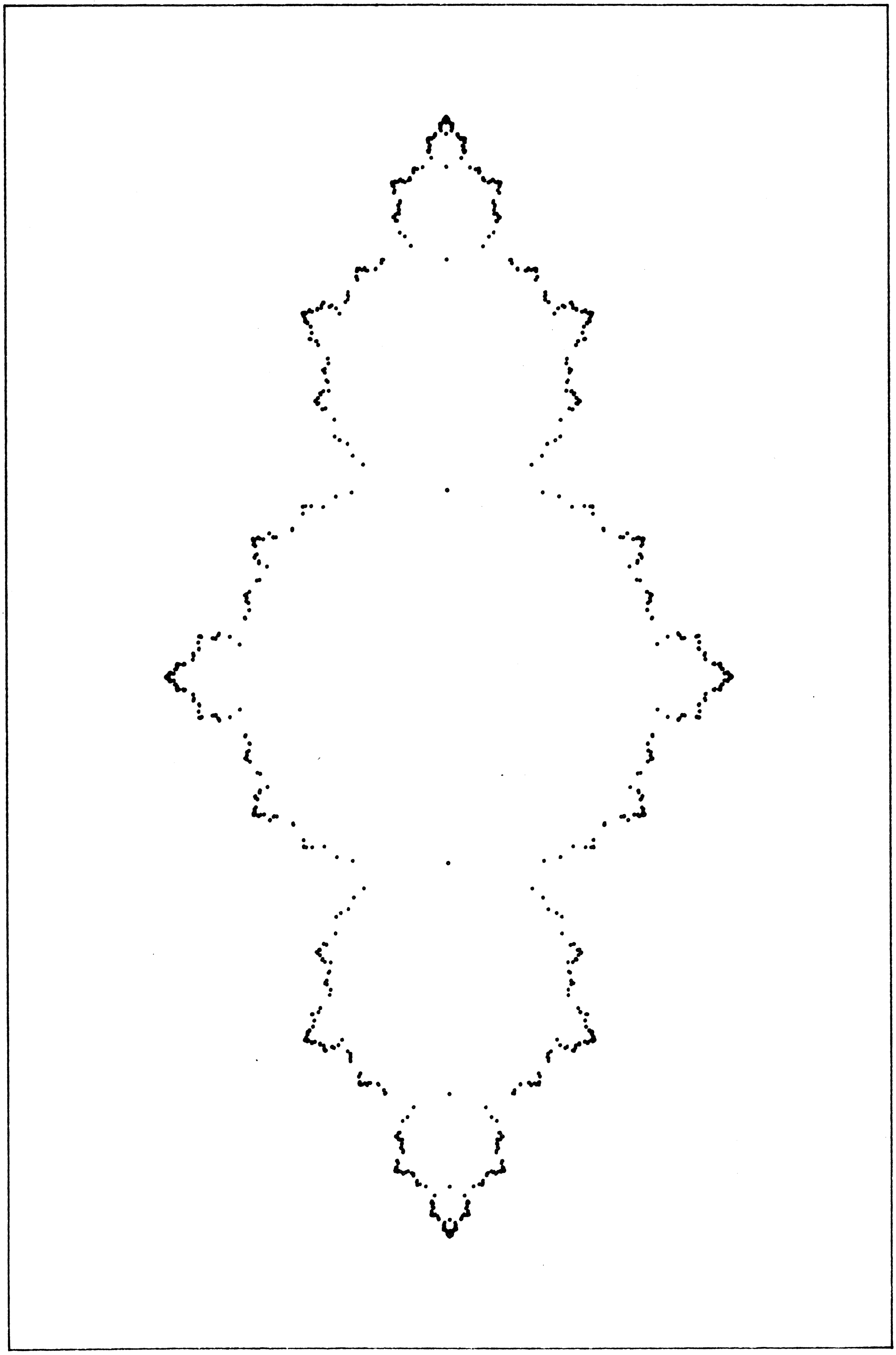

Figure 10. The San Marco attractor of $az(1-z) \to z$.

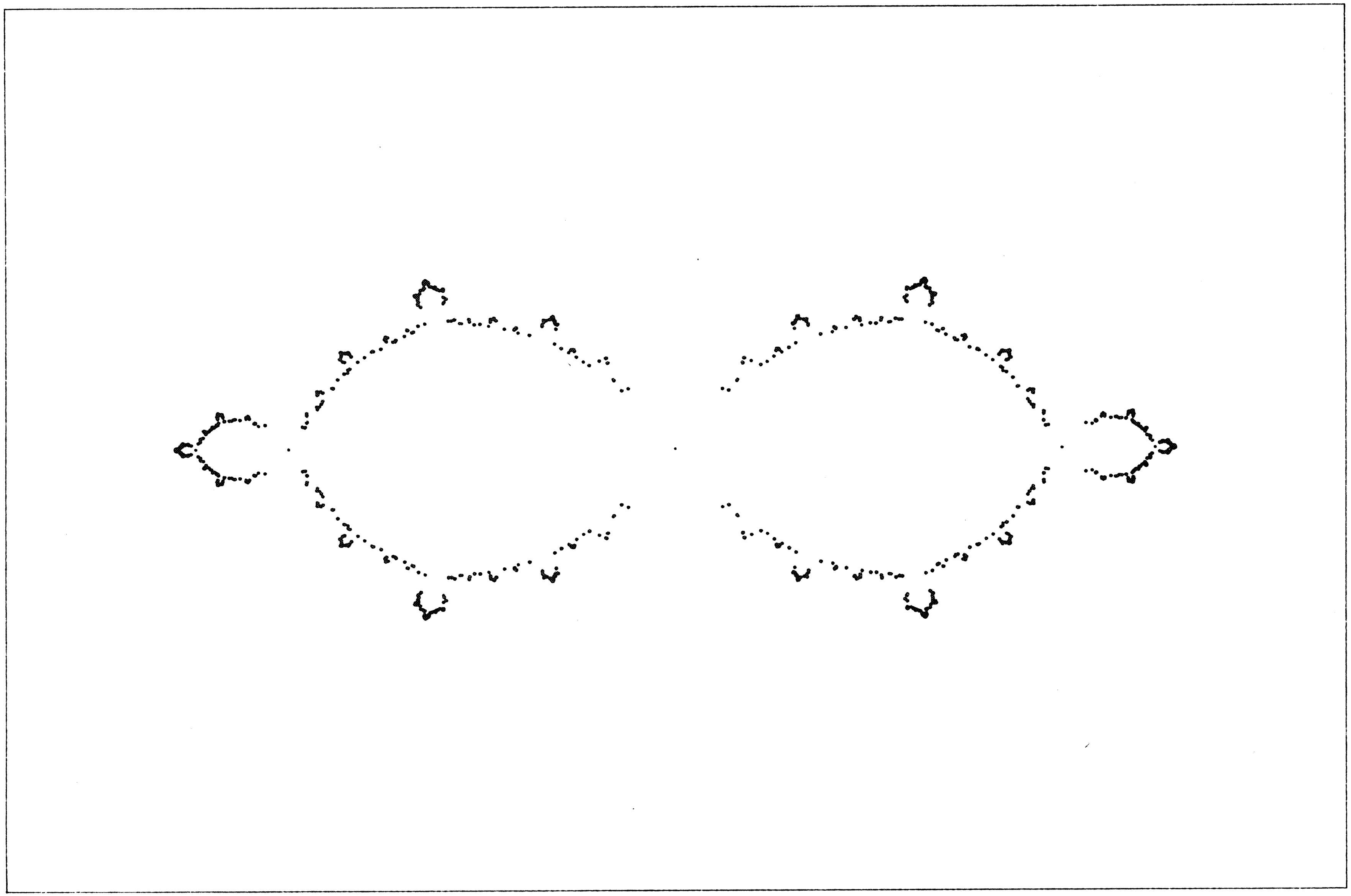

Figure 11. Loops in loops in loops in ...,
The Julia set of $z \to (3z-z^3)/2$.

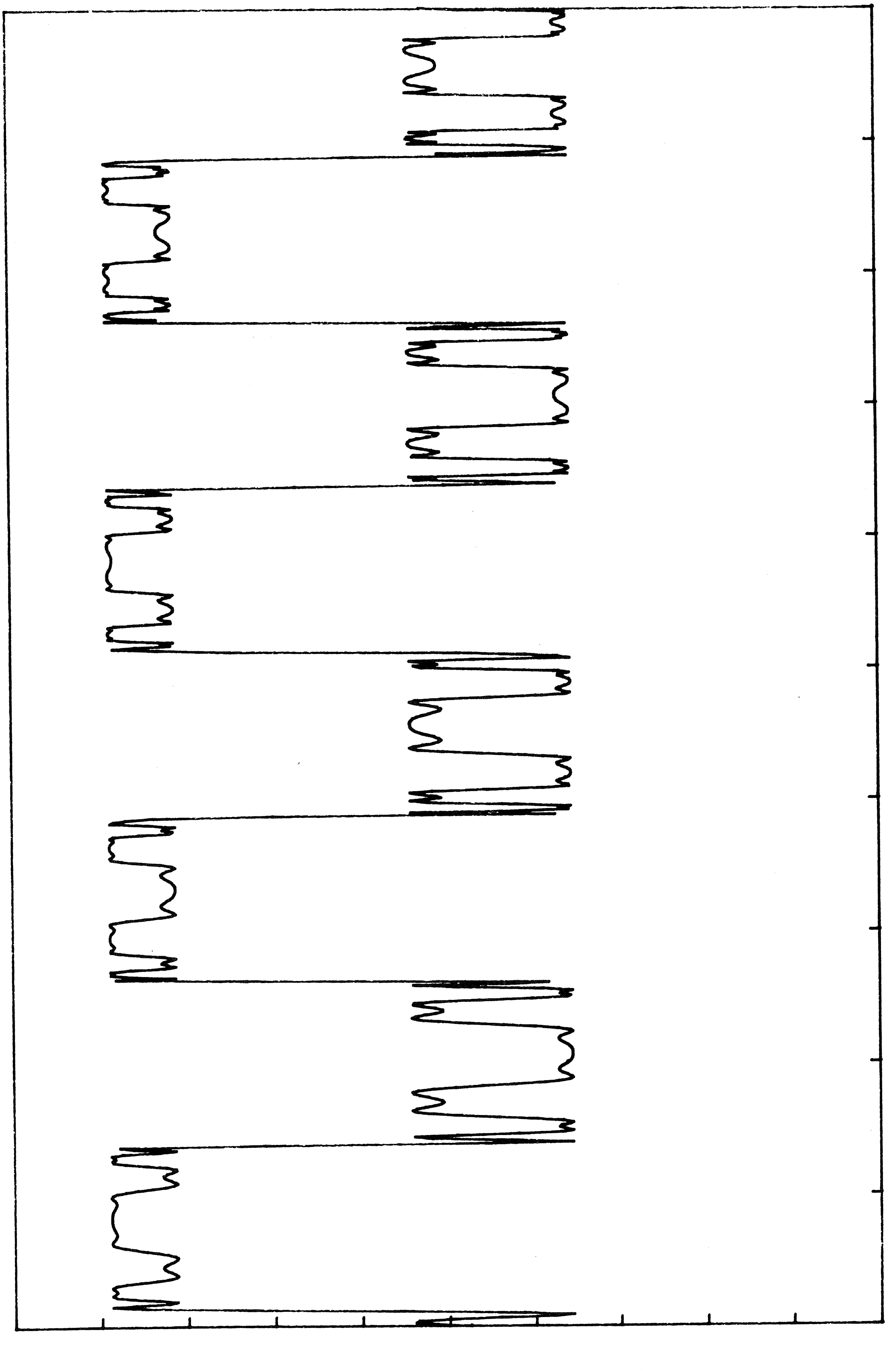

Figure 12. "Almost periodic" behaviour of an entire function on the positive real axis.

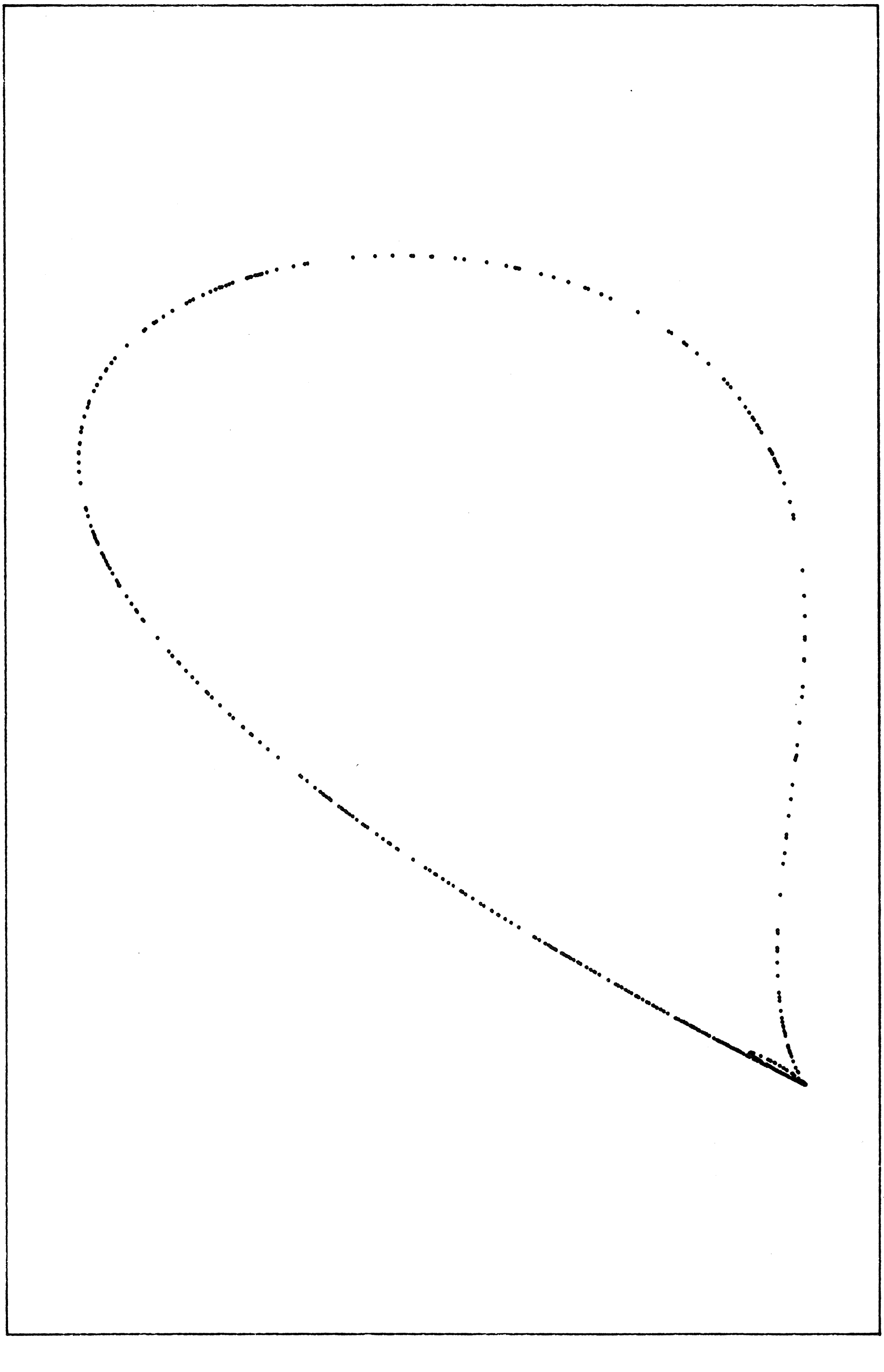

Figure 13. Strange attractor of $x_{n+1} = ax_n(1-x_{n-1})$ for a = 2.27.

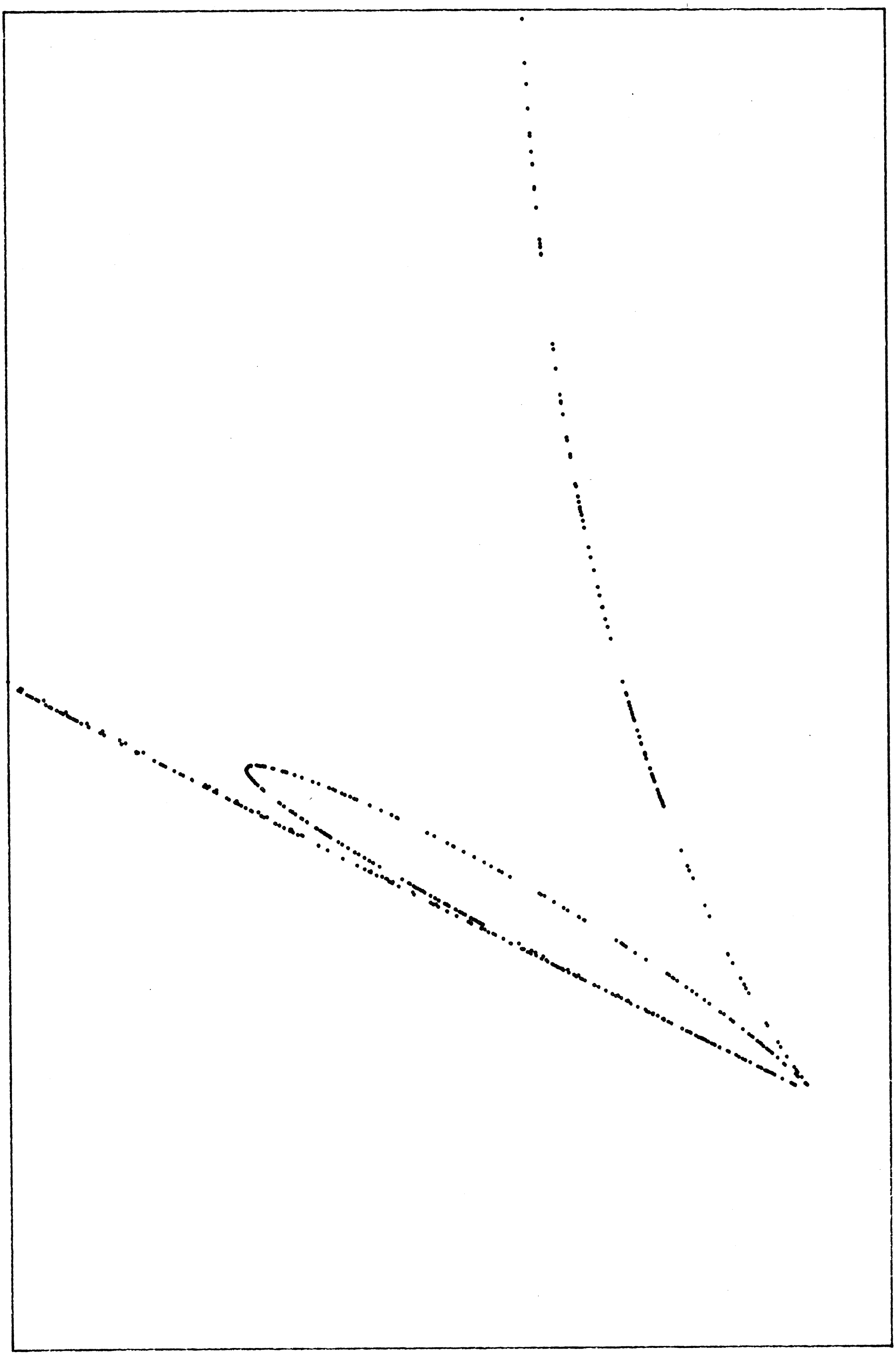

Figure 14. Blow-up of fig. 13.

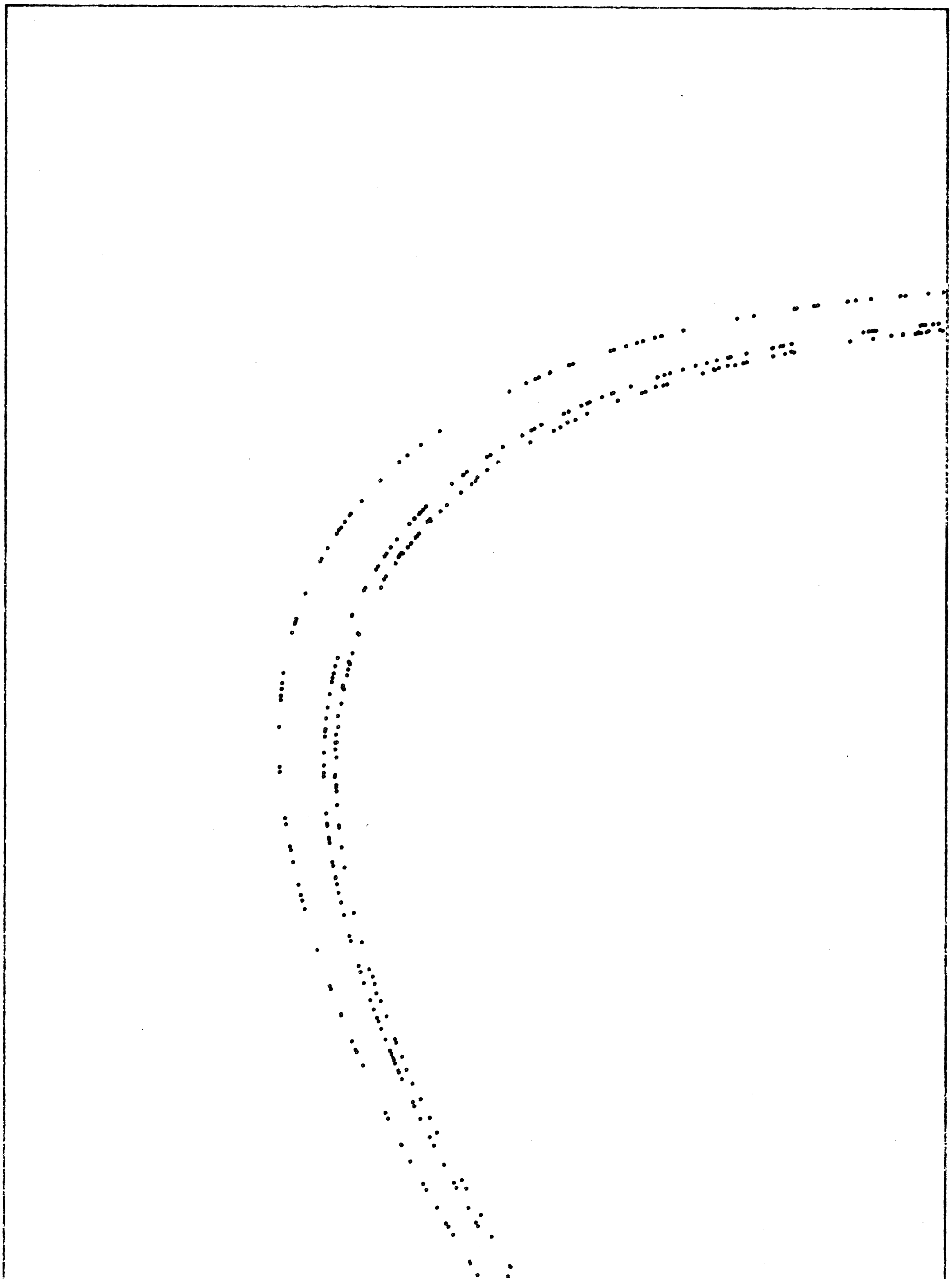

Figure 15. Blow-up of fig. 14.

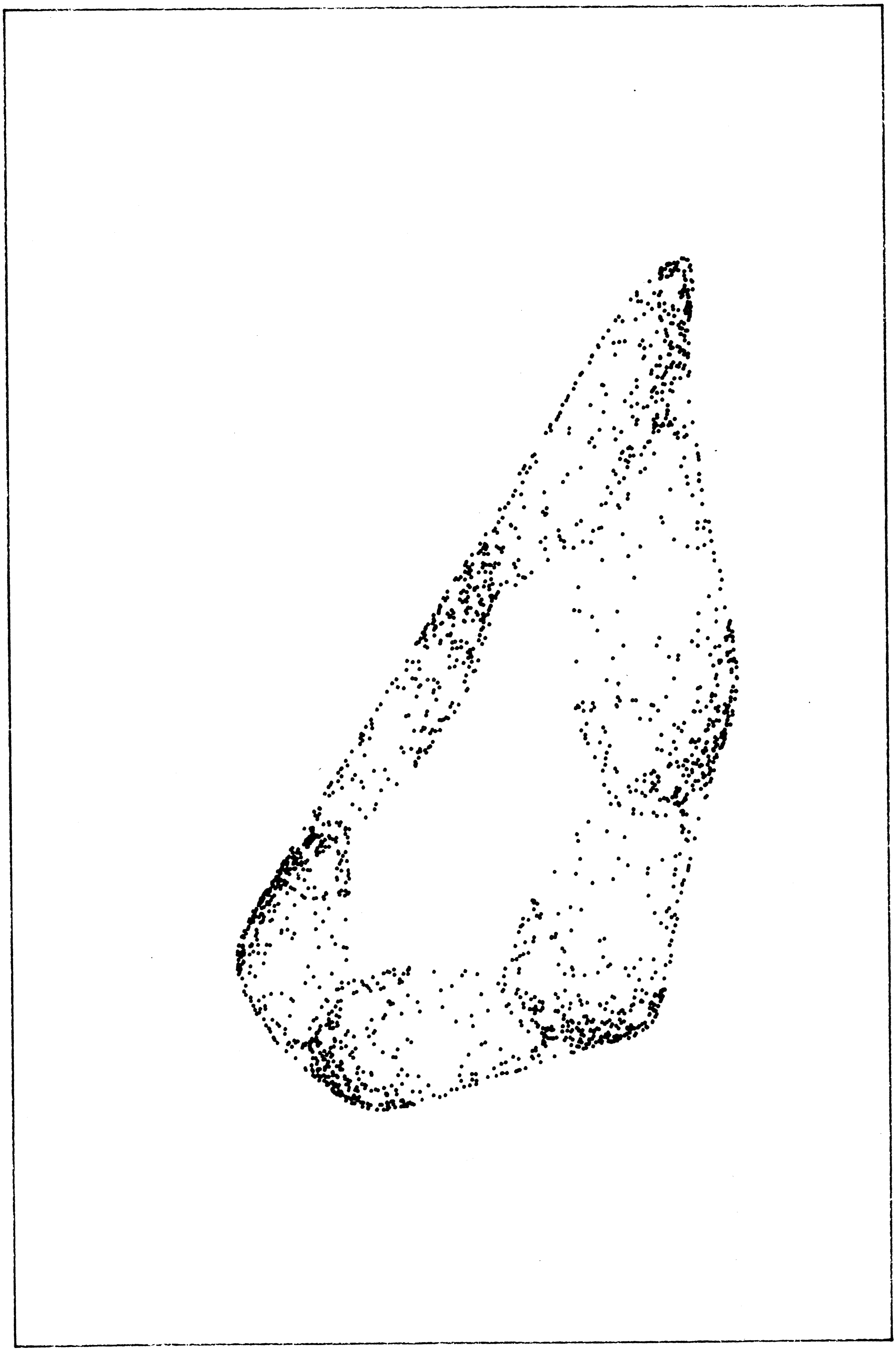

Figure 16. Strange attractor of a predator-pray model.